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Cumulative Diminuations with Fibonacci Approach, Golden Section and Physics

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Abstract In this study, physical quantities of a nonequilibrium system in the stages of its orientation towards equilibrium has been formulated by a simple cumulative diminuation mechanism and Fibonacci recursion approximation. Fibonacci p-numbers are obtained in power law forms and generalized diminuation sections are related to diminuation percents. The consequences of the fractal structure of space and the memory effects are concretely established by a simple mechanism. Thus, the reality why nature prefers power laws rather than exponentials ones is explained. It has been introduced that, Fibonacci p-numbers are elements of a Generalized Cantor set. The fractal dimensions of the Generalized Cantor sets have been obtained by different methods. The generalized golden section which was used by M.S. El Naschie in his works on high energy physics is evaluated in this frame.

1 Introduction

In the formation of self-similar life forms nature prefers certain sections. Most important one of those sections is named as divine section which is $\phi = 0.618$. In this preference of nature, it is accustomed to wisdom and scientific thought that nature takes care of the outcoming achievement to be sound, economical, maintainable and esthetic. From historical developments point of view; it is observed that human beings as a part of nature, have been using golden section in a harmonious manner with nature in architecture and art works [1–5].

Nowadays progressing researches expose that golden section is encountered in a wide spectrum starting from deoxyribonucleicacid (DNA) which encodes the genetic inheritance of human beings, extending to particle physics [6–8].

In this study, creation of the golden section is put forward by a dynamical mechanism of a cumulative diminuation. More over, it is shown that golden ϕ is closely connected with the cumulative progress of the processes and ϕ is related to the reduction rate λ. On the other hand, a relation with λ and the fractal dimension of the process is established. In ordinary approaches, which are non-Markovian with Euclidean geometry, the fact that the process

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possesses long range memory effect and evolves in a fractal geometry have not been taken into account. In our opinion, as a consequence of this approach, the effect which arises in cumulative diminuation, does not take the place it deserves in numerical systems in other words in mathematical descriptions. Here in our approach, the fractal structure of space and the long range memory effect exhibit themselves naturally as a cumulative diminuation effect. Our point of view is that one of the most appropriate number systems is Fibonacci p-type numbers. From this angle of vision, we have the conviction that harmony mathematics which is constructed by number systems which take into account the fractal property of space and memory effect is a necessary and a natural instrument in the description of the evolution of physical phenomena [6, 8]. Therefore, distribution functions which are recognized in exponential mathematical forms should be in power forms [9–11].

In Sect. 2 of this work, the diminuation mechanism of the model process is brought up by Fibonacci approach and the mathematical formulation is carried out. In Sect. 3, the physical origin of the q-entropy index which was introduced in nonextensive statistical mechanics is explained. In Sect. 4, Generalized Cantor set whose elements are constructed from the powers of cumulative generalized section is introduced. In Sect. 5, the fractal dimensions of the Generalized Cantor sets are obtained. In Sect. 6, the fractal dimension calculations are carried out using another method and Pisot-Vijayvaraghavan (P-V) number is calculated. At the end of the work, conclusions and discussions are given.

2 Mathematical Formulation of Cumulative Diminuation

Cumulative diminuations are met in our daily life and examples may be given as extinguishing of a fire, lowering of the value of currency with inflation, decreasing of a population and disintegration of a rock.

With the purpose of putting forward the mechanism of cumulative diminuation let us consider a system which diminishes in the course of time. This study might be considered as a reverse process of the historical Fibonacci's rabbit generation problem. In other words, in the course of time population of a certain number of rabbits decreases.

As an example to our approach, let the quantity a which is possessed at the beginning of the diminuation, disintegrate cumulatively with a reduction rate λ. Let the quantities which the amount a will attain later, at time intervals $0, \Delta t, 2\Delta t, \dots, n\Delta t$ be $A_0, A_1, A_2, \dots, A_n$ respectively. Now we define cumulative diminuation for a p-process such that; the amount possessed at the rth step is obtained by subtracting the disintegration amount in the $(r-1)$st step (i.e. $a\Delta t\lambda_p$) from the amount at the $(r-1)$st step. With this mechanism, quantities at $r = 0, 1, 2, \dots, n$th steps are obtained in the following form:

$$\begin{array}{ll} 0 & A_0 = a, \\ \Delta t & A_1 = (1-\lambda_p \Delta t)a, \\ 2\Delta t & A_2 = (1-\lambda_p \Delta t)^2 a, \\ \vdots & \vdots \\ r\Delta t & A_r = A_{r-1} - \lambda_p \Delta t A_{r-1} = (1-\lambda_p \Delta t)^r a, \\ \vdots & \vdots \\ n\Delta t & A_n = (1-\lambda_p \Delta t)^n a. \end{array} \tag{1}$$

Generally, reduction rate for a p-process could be defined as

$$\lambda_p = -\frac{\Delta A}{A \Delta t}, \quad 0 \le \lambda_p \le 1 \tag{2}$$

which is the disintegration percent per unit time. Taking into account the fact that for a process with a duration time t, diminuation time interval is

$$\Delta t = \frac{t}{r}, \tag{3}$$

then generally for the rth step of the process

$$A_r(t) = a\left(1 - \lambda_p \frac{t}{r}\right)^r \tag{4}$$

could be obtained. It is seen that diminuation is an expression in power law form. If t takes very large values, using the definition of number e

$$A(t) = ae^{-\lambda_p t} \tag{5}$$

is obtained. This result could also be obtained by following a different way. Starting from the assumption that change in A presented in (1) is proportional to A and dt one could write

$$dA = -\lambda_p A dt. \tag{6}$$

Minus sign on the right hand side of (6) shows that A decreases in the course of time in other words the process is a reduction. A solution to (6) is

$$A(t) = e^{-\lambda_p t} e^{\ln C} \tag{7}$$

where C is an integration constant. If the initial condition $A = a$ for $t = 0$ is taken into account A is calculated to be,

$$A(t) = ae^{-\lambda_p t} \tag{8}$$

and this is the solution of (6). Therefore, one could say that (8) is valid for the cases where total time t of the process is large enough that is to say when the system exhibits asymptotic behavior.

For deviations from the asymptotic behavior, using the demonstration obtained by C. Tsallis in nonextensive statistical mechanics [9], the interval $0 \le n \le \infty$ could be squeezed to a more narrow interval. For this purpose let us take

$$n = \frac{1}{q-1}. \tag{9}$$

It is obvious that for $q \to 1$, $n \to \infty$ and for $q = 2$, $n \to 1$. In this case, (4) could be written as

$$A_{\frac{1}{q-1}}(t) = a[1 - (q-1)\lambda_p t]^{\frac{1}{q-1}}; \quad 1 \le q \le 2 \tag{10}$$

or

$$A_{\frac{1}{q-1}} = e_q(-\lambda_p t). \tag{11}$$

For $q \to 1$, (10) and (11) are identical to (5);

$$A_\infty(t) = ae_1(-\lambda_p t). \tag{12}$$

The results achieved here are in one respect closely related to the definition of the logarithm of x and according to our investigations is equivalent to the representation

$$\ln x = \lim_{n\to\infty} \frac{x^{\frac{1}{n}} - 1}{\frac{1}{n}}. \tag{13}$$

Thus, by taking into account the definition of the logarithm of a number given by (13) one could express correctly the diminishing processes of cumulatively diminuating systems.

Keeping in mind this definition and after solving (6) one ends up with:

$$A_n = a\left(1 - \frac{\lambda_p t}{n}\right)^n. \tag{14}$$

In this study, to indicate the importance of the variation mechanism in cumulative diminuation processes $dA \propto A$ is taken. In more complex processes where dA is proportional to the powers of A, solution of (6) becomes Mittag-Leffler function.

The power form of the expression in (4) which gives the number of steps in the process, originates from the nature of the cumulative diminuation mechanism. With its recurrence characteristics, (4) carries the genetic effects of the past on the other hand from the point of view of extending over a wide time interval it covers in itself long range memory effect. We call this common effect as cumulative effect. If cumulative effect is taken aside by choosing $\Delta t = 1$ it is obvious that

$$A_0 = a, \quad A_1 = a - \lambda_p a, \quad A_2 = a - 2\lambda_p a, \quad \ldots, \quad A_{n-1} = a - n\lambda_p a. \tag{15}$$

One could say that, in cumulative diminuation, the cumulative effect leads to an excessive exhaustion of the system under investigation. It is seen that, in order to keep the system under control, the factors which slow down the process or resist to cumulative diminuation are important. Since it is essential that the system must be long sustainable, introduction of the parameters which reduce the cumulative effect is a necessity.

In order to exhibit the additive influence of the cumulative effect explicitly, sum of the amounts reached at the steps 0, $\Delta t, 2\Delta t, \ldots, n\Delta t$ would be more meaningful. Therefore, let us write the sum

$$T = \sum_{r=0}^{n} (1 - \lambda_p \Delta t)^r a. \tag{16}$$

Since the sequence in (16) is a finite-geometrical sum, one could perform the summation leading to

$$T = \frac{1 - (1 - \lambda_p \Delta t)^{n+1}}{\lambda_p \Delta t} a \tag{17}$$

where $(1 - \lambda_p \Delta t)$ is a common factor.

In the $(1 - \lambda_p \Delta t)^{n+1} \ll 1$ case, in other words for $\lambda_p \Delta t \to 1$ one gets

$$T = \frac{a}{\lambda_p \Delta t}. \tag{18}$$

That is to say $T \to a$, namely remains relatively identical. In the case of simple diminuation where $\lambda_p \ll 1$, the total amount is found as

$$T = \sum_{r=0}^{n} (a - r\lambda_p a) = (n+1)a\left(1 - \frac{n}{2}\lambda_p\right). \tag{19}$$

A comparison of (17) and (19) clearly exhibits the cumulative effect.

3 Relation of Cumulative Diminuation with Thermostatistics

According to our evaluations, the laws of physics obtained using Euclidean geometry which is not appropriate for describing the nature and by non-Markovian approaches, remain inadequate.

Consequently scientists incline towards new explorations. C. Tsallis' generalized entropy and his generalization of Boltzmann distributions could be evaluated in this content. C. Tsallis generalized the entropy and with this point of departure obtained the distribution functions. Tsallis pointed out that the entropy index that he exposed in his works was related to the fractal structure of space and to the long range memory effect of the phenomena. Authors of the present study, starting from Tsallis' approach, generalized the entropy and distribution functions belonging to quantum systems [10, 11]. Other workers have conducted research on the applications of generalized quantum distributions but here we are in a position to cite only some of them [12–15].

In this study the physical origin of the entropy index q (which is introduced in nonextensive thermostatistics) is put forward by proposing a equitime interval evolution of a process in a fractal space in a consecutive manner and by putting ahead the cumulative diminuation effect. In this context, taking into consideration (9) one could say that, the entropy index q shows the number of steps, a nonequilibrium physical system undergoes while tending to equilibrium after interacting with the environment and $q = 1$ corresponds to the state of equilibrium of the system. Here, by pursuing the Fibonacci recurrence approach long range genetic effect is introduced and by the discrete evaluation of the phenomena, the fractal property of the space has been taken into consideration.

4 Cumulative Diminuation and Generalized Section Scale

When calculating the fractal dimension in a cumulative diminuation process, it is instructive to establish a relation between fractal geometry, cumulative diminuation mechanism and generalized golden section. In the generalization of golden section, a segment AB is divided by a point C using the section

$$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p \tag{20}$$

where $p = 0, 1, 2, \ldots$.

Generalized golden section problem could algebraically be reduced to [6]

$$\tau_p^{p+1} = \tau_p^p + 1 \tag{21}$$

where τ_p is a positive root of (21).

Although Fibonacci numbers are generally attributed to growing systems, it is obvious that this approach could also be applied to diminishing systems.

With the diminishing mechanism we have mentioned before, Fibonacci p-numbers diminish with respect to the power law expression:

$$F_p(n) = (1-\lambda_p)^n a \tag{22}$$

particularly at the further steps of the process. Thus, Fibonacci p-numbers set forms a set with elements

$$\{a, (1-\lambda_p)a, (1-\lambda_p)^2 a, \ldots, (1-\lambda_p)^n a\} \tag{23}$$

for a cumulatively diminishing system. If

$$a = 1 \quad \text{and} \quad \phi_p = 1-\lambda_p = \frac{1}{\tau_p} \tag{24}$$

is taken, then expression (23) becomes

$$\{1, \phi_p, \phi_p^2, \phi_p^3, \ldots, \phi_p^n\}. \tag{25}$$

This set could be named as "Generalized Cantor Set".

5 Calculation of the Fractal Dimension of the Generalized Cantor Set

Decreasing Fibonacci p-numbers describe a cumulative diminuation process. Let us apply the cumulative diminuation mechanism presented in Sect. 3 to the diminishing of a length expressed by a. That is to say, let us assume that the amount of length A_{n-1} which is owned at the step $(n-1)$ diminishes (reduces) by an amount $\lambda_p A_{n-1}$ in order to establish the amount A_n at step n. Thus elements of the set obtained by cumulative diminuation could be written as

$$\begin{aligned}
&A_0 = a, \\
&A_1 = a - \lambda_p a = a(1-\lambda_p) = a - L_1; \quad L_1 = \lambda_p L_0, \\
&A_2 = A_1 - \lambda_p A_1 = A_1 + L_1 - \lambda_p L_1 = a(1-\lambda_p)^2; \quad L_2 = \lambda_p L_1, \\
&\vdots \\
&A_n = A_{n-1} - \lambda_p A_{n-1} = A_{n-1} + L_{n-1} - \lambda_p L_{n-1}; \quad L_n = \lambda_p L_{n-1}.
\end{aligned} \tag{26}$$

It follows that in the present case, A_n at the nth step is attained by adding L_{n-1} to A_{n-1} of the $(n-1)$st step and then a subtraction $\lambda_p L_{n-1}$ succeeds.

According to Mandelbrot's definition one could write

$$\begin{aligned}
A_{n+1} &= \text{const} L_{n+1}^{d-D_p}, \\
A_n &= \text{const} L_n^{d-D_p}
\end{aligned} \tag{27}$$

where d is the dimension of measure, while D_p is the dimension of Hausdorff-Besicovitch dimension. Then from (23) the following expression could be written down:

$$\frac{A_{n+1}}{A_n} = \frac{(1-\lambda_p)^{n+1}}{(1-\lambda_p)^n}; \qquad \frac{L_{n+1}}{L_n} = \lambda_p. \tag{28}$$

By substituting (28) into (27) one could find out the generalized Hausdorff-Besicovitch dimension of the measurement of the system:

$$D_p = d - \frac{\ln(1-\lambda_p)}{\ln \lambda_p}. \tag{29}$$

This dimension is the dimension of the set which is given by (23) and (25). From (29) dimensions for Fibonacci p-type Cantor sets, are numerically given in the following forms:

$$\begin{aligned}
&\text{for } p=0, && \phi_0 = 1-\lambda_0 = \frac{1}{2}, && \lambda_0 = \frac{1}{2} = 0.5, && D_0 = 0,\\
&\text{for } p=1, && \phi_1 = 1-\lambda_1 = \frac{1}{1.618}, && \lambda_1 = 0.382, && D_1 = 0.5,\\
&\text{for } p=2, && \phi_2 = 1-\lambda_2 = \frac{1}{1.465}, && \lambda_2 = 0.317, && D_2 = 0.668,\\
&\text{for } p=3, && \phi_3 = 1-\lambda_3 = \frac{1}{1.379}, && \lambda_3 = 0.275, && D_3 = 0.751,\\
&\text{for } p=4, && \phi_4 = 1-\lambda_4 = \frac{1}{1.321}, && \lambda_4 = 0.245, && D_4 = 0.801,\\
&\dots\\
&\text{for } p=\infty, && \phi_\infty = 1-\lambda_\infty = 1, && \lambda_\infty = 0, && D_\infty = 1.
\end{aligned} \tag{30}$$

Here we have used ϕ_p sections given in Ref. [6].

As an application let us calculate the H-D dimension of the classical triadic Cantor set in the same manner. Since the element a of the set diminishes with the section $\lambda = \frac{1}{3}$ one obtains:

$$D = 1 - \frac{\ln(1-\frac{1}{3})}{\ln\frac{1}{3}} = \frac{\ln 2}{\ln 3}. \tag{31}$$

Thus, classical Cantor set is a set which has a disintegration constant $\lambda = \frac{1}{3}$ and has the elements $\{1, \phi, \phi^2, \dots, \phi^{n-1}\}$. In other words, it is a set whose elements are $\{(1-\frac{1}{3})a, (1-\frac{1}{3})^2 a, \dots, (1-\frac{1}{3})^{n-1}a\}$. Here in this case ϕ is taken as

$$\phi = 1 - \frac{1}{3} = 1 - \lambda.$$

What calls ones attention here is that; in the generalized golden section one obtains a rational number of the form $\frac{1}{\tau_1} = \phi_1 = 1 - \lambda_1 D_1 = 0.5$. Whereas now the diminishing percent is $\lambda_1 = 0.382$.

Owing to a theorem sated by Mauldin-Williams, with the probability of one, a randomly constructed triadic Cantor set has the Hausdorff-Besicovitch dimension equal to the golden section. With the method developed in this study, the diminishing section λ_m of the concomitant set related to this dimension, could be calculated using (29). Without proceeding into the calculations one could state that $0.317 < \lambda_m < 0.382$. For the section ϕ_m on the other hand, one could see that it is in between the values $\phi_1 = 0.618$ corresponding to $p = 1$ and $\phi_2 = 0.682$ corresponding to $p = 2$.

As it could be recognized, starting with a quantity a and passing through a continuous cumulative diminuation process, one ends up with a Cantor dust as a result. One notices that decreasing in the diminishing percent (λ_p) leads to an increase in the dimension. However this is expected as a natural conclusion of the fractal property of space. Meanwhile this is a result which determines the structure.

6 Cumulative Diminuation, Generalized Sections and Physical Applications

In M.S. El Naschie's works on high energy physics and electromagnetic weak interactions golden section ϕ_1 plays a very important role [16–21]. Let us calculate the averages of the powers of the set elements of the Cantor set we have obtained by our generalized sections and also calculate the mean square of the powers of the set elements. From the random fluctuation theory we know that in a random set $\langle n_p^2\rangle$ (not $\langle n_p\rangle$) possesses a physical importance and it gives the dimension of the space-time it represents. In this connection let us return to the calculation of the generalized Pisot-Vijayvaraghavan (PV) number.

By taking into consideration this set some of the quantities could have physical correspondence. First of all, the elements of the set gives the total population. Thus, the population of a cumulatively diminishing assembly becomes

$$Z_p = 1 + \phi_p + \cdots + \phi_p^n + \cdots. \tag{32}$$

Since (32) is a geometrical series whose common factor is ϕ_p ($\phi_p < 1$), one obtains

$$Z_p = \frac{1}{1-\phi_p} \tag{33}$$

where $n \to \infty$ and $\phi_p^n \to 0$ is taken.

As a special case if $p = 1$ is chosen one gets

$$Z_1 = \frac{1}{\phi_1^2} = (1+\phi_1)^2 = 2 + \phi_1 = 2.618 \tag{34}$$

where we have used the following properties of the golden section:

$$\phi_1 = \frac{1}{1+\phi_1} \quad \text{and} \quad 1 - \phi_1 = \phi_1^2. \tag{35}$$

On the other hand, for the generalized Cantor set,

$$S_p(n) = \sum_{n=0}^{\infty} n\phi_p^n = \phi_p + 2\phi_p^2 + \cdots + n\phi_p^n + \cdots \tag{36}$$

could give an idea about the total weight which shows the step number weighed total. This in turn is calculated to be

$$S_p(n) = \phi_p \frac{\partial Z_p}{\partial \phi_p} = \phi_p \frac{\partial}{\partial \phi_p} \frac{1}{1-\phi_p} = \frac{\phi_p}{(1-\phi_p)^2}. \tag{37}$$

When $p = 1$ total S_1 for the golden section is obtained:

$$\begin{aligned} S_1 &= \frac{\phi_1}{(1-\phi_1)^2} = \frac{\phi_1}{\phi_1^4} = \frac{1}{\phi_1^3} = (1+\phi_1)(1+\phi_1)(1+\phi_1) = (1+\phi_1)(1+2\phi_1+\phi_1^2) \\ &= (1+\phi_1)(1+2\phi_1+1-\phi_1) = (1+\phi_1)(2+\phi_1) = 1+3\phi_1^2+3\phi_1+\phi_1^3 \end{aligned}$$

$$
\begin{aligned}
&= 1 + 3(1-\phi_1) + 3\phi_1 + \phi_1^3 = 1 + 3 - 3\phi_1 + 3\phi_1 + \phi_1^3 = 4 + \phi_1^3 \\
&= 4.236067977.
\end{aligned} \tag{38}
$$

Taking into account (33) and (37)

$$
n_p = \frac{S_p(n)}{Z_p} = \frac{1}{1-\phi_p} = \frac{1}{\lambda_p} \tag{39}
$$

is found. For the special case $p = 1$ one arrives to

$$
n_1 = \frac{1}{\phi_1^2} = 2 + \phi_1 = \frac{1}{\lambda_1} = 2.618. \tag{40}
$$

From (39) for the values

$$
(p = 0, \phi_0), \quad (p = 1, \phi_1), \quad (p = 2, \phi_2), \quad \ldots, \quad (p = \infty, \phi_\infty)
$$

one gets

$$
\begin{aligned}
&n_0 = 2, \quad n_1 = 2.618, \quad n_2 = 3.155, \\
&n_3 = 3.636, \quad n_4 = 4.082, \quad \ldots, \quad n_\infty = \infty.
\end{aligned} \tag{41}
$$

Following M.S. El Naschie [16], let us calculate the fractal dimension D_p^c for the generalized Cantor set:

$$
S_p(n^2) = \sum_{n=0}^{\infty} n^2 \phi_p^n = \phi_p \frac{\partial}{\partial \phi_p} S_p(n) \tag{42}
$$

could be written down. On the other hand taking the value given in (37) for $S_p(n)$ one ends up with:

$$
S_p(n^2) = \phi_p \frac{1+\phi_p}{(1-\phi_p)^3}. \tag{43}
$$

For the generalized Cantor set, taking into account (32) and (35), the fractal dimension is obtained as

$$
D_p^c = \frac{S_p(n^2)}{S_p(n)} = \frac{1+\phi_p}{1-\phi_p}.
$$

The dimensions corresponding to

$$
\begin{aligned}
&(p = 0, \phi_0 = 0.5), \quad (p = 1, \phi_1 = 0.618), \quad (p = 2\phi_2 = 0.683), \quad (p = 3, \phi_3 = 0.747), \\
&(p = 4, \phi_4 = 0.755), \quad \ldots, \quad (p = \infty, \phi_\infty = 1)
\end{aligned}
$$

of a p-type Cantor set are respectively found to be:

$$
D_0^c = 3, \quad D_1^c = 4.236, \quad D_2^c = 5.309, \quad D_3^c = 6.905, \quad D_4^c = 7.164, \quad \ldots, \quad D_\infty^c = \infty.
$$

In the special case $p = 1$, the fractal dimension is the (PV) number [16] which is given by

$$
D_1^c = \frac{1+\phi_1}{1-\phi_1} = 4 + \phi_1^3 = 4.236\ldots. \tag{44}
$$

7 Conclusions and Discussions

The inadequacy of the expressions describing the physical phenomena which are obtained as the results of works in Euclidean space and non-Markovian approaches directed the physicists towards new searches. Therefore, in this content, the fractal property of space and the long range memory effect have been taken into account on the course of successive evaluation of the processes.

Cantor set has been established with the segments obtained in the process. The existence of the golden section, which is encountered starting from high energy physics to other branches of science and also art, is proposed by a straight forward mechanism and this section is related to diminishing percent of the process. Moreover, in this study it is shown that by a simple dynamical diminuation (disintegration) mechanism and with the generalized diminishing sections, for the first time, to our knowledge, Generalized Cantor set is introduced. Generalized fractal dimensions for Generalized Cantor sets are obtained in terms of reduction rates. It is also demonstrated that, well known PV number is the dimension of a special case of the set.

Furthermore, with the simple diminuation mechanism, the physical origin of the entropy index q encountered in nonextensive thermostatistics has been clarified. On mathematical grounds, it is established that as a result of cumulative effect, laws expressed in exponential forms should be in power law forms. It is concluded that the cumulative diminuation effects take into account the memory effect and the fractal property of space.

References

1. Atalay, B.: Math and Monalisa. Smithsonian Books, Washington (2004)
2. Feder, J.: Fractals. Plenum, New York (1988)
3. Kak, S.: The Golden Mean and the physics of aesthetics. Arch. Phys. (2004), physics/0411195
4. Stakhov, A.P.: The Golden Section, secret of egyptian civilization and harmony mathematics. Chaos Solitons Fractals **30**, 490–505 (2006)
5. Hejazi, M.: Geometry in nature and Persian architecture. Build. Environ. **40**(10), 401–406 (2005)
6. Stakhov, A.P.: The generalized principle of the Golden Section and its applications in mathematics, science and engineering. Chaos Solitons Fractals **26**(2), 263–289 (2005)
7. Stakhov, A.P.: Fundementals of a new kind of mathematics based on the Golden Section. Chaos Solitons Fractals **27**, 1124–1146 (2006)
8. Stakhov, A.P., Rozin, B.: The continuous function for the Fibonacci and Lucas numbers. Chaos Solitons Fractals **28**, 1014–1025 (2006)
9. Tsallis, C.: Possible generalization of Boltzmann-Gibbs statistics. J. Stat. Phys. **52**, 479–487 (1988), corrigenda: Curado, E.M.F., Tsallis, C.: Generalized statistical mechanis: connection with thermodynamics. J. Phys. A **24**, L69–L72 (1991); **24**, 3187 (1991); **25**, 1019 (1992)
10. Büyükkılıç, F., Demirhan, D.: A fractal approach to entropy and distribution functions. Phys. Lett. A **181**, 24–28 (1993)
11. Büyükkılıç, F., Demirhan, D., Güleç, A.: A statistical mechanical approach to generalized statistics of quantum and classical gases. Phys. Lett. A **197**, 209 (1995)
12. Teweldeberhan, A.M., Plastino, A.R., Miller, H.G.: On the cut-off prescriptions associated with power-law generalized thermostatistics. Phys. Lett. A **343**, 71–78 (2005)
13. Plastino, A.R., Plastino, A., Miller, H.G., Uys, H.: Foundations of nonextensive statistical mechanics and its cosmological applications. Astrophys. Space Sci. **290**, 275–286 (2004)
14. Wang, Q.P.A., Mehaute, A.L.: On the generalized distribution functions of quantum gases. Phys. Lett. A **235**, 222–226 (1997)
15. Wang, Q.P.A., Mehaute, A.L.: On the generalized blackbody distribution. Phys. Lett. A **237**(1–2), 28–32 (1997)
16. El Naschie, M.S.: Complex vacuum fluctuation as a chaotic "limit" set of any Klenian group transformation and the mass spectrum of high energy particle physics via spontaneous scef-organization. Chaos Solitons Fractals **17**, 631–638 (2003)

17. El Naschie, M.S.: From atomic to sperstring scale in a braided Cantorian $\varepsilon^{(\infty)}$ space. Chaos Solitons Fractals **12**, 1439–1448 (2001)
18. El Naschie, M.S.: Superstrings, knots and noncommutative geometry in $\varepsilon^{(\infty)}$ space. Int. J. Theor. Phys. **37**(14), 8 (1998)
19. El Naschie, M.S.: Remarks to the PV number $(\frac{\sqrt{5}+1}{2})^3 = 4.326$. Chaos Solitons Fractals **10**(8), 1335–1341 (1999)
20. El Naschie, M.S.: Elemantary number theory in superstrings, loop quantum mechanics, twistor and E-infinity high energy physics. Chaos Solitons Fractals **27**, 297–330 (2006)
21. El Naschie, M.S.: On a class of general theories for high energy particle physics. Chaos Solitons Fractals **14**, 649–668 (2002)